On Nonbondage Numbers of a Graph

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Abstract
For a graph $G=(V, E)$, a set $D\subseteq V$ is a dominating set if every vertex in $V – D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a total dominating set. The nonbondage number $b_n(G)$ of G is the maximum cardinality among all sets of edges $X\subseteq E$ such that $\gamma(G – X) = \gamma(G)$. A set $D\subseteq V$ is a strong dominating set if every vertex in $V – D$ has a neighbor u in D such that the degree of u is not smaller than the degree of v. The strong domination number $\gamma_s(G)$ of G is minimum cardinality of a strong dominating set. The strong nonbondage number $b_{sn}(G)$ of a nonempty graph G is the maximum cardinality among all sets of edges $X\subseteq E$ such that $\gamma_s(G – X) =\gamma_s(G)$. In this paper, some results on the nonbondage number, exact values of $b_n(G)$ for some standard graphs are obtained. Also some results on the strong nonbondage number and bondage number are established. Also Nordhaus-Gaddum type results are found.

Keywords
Bondage Number, Nonbondage Number, Strong Nonbondage Number, Chromatic Number, Connectivity
Mathematics Subject Classification: *05C.*

I. Introduction

All graphs considered here are finite, undirected, without loops or multiple edges and isolated vertices and have p vertices and q edges. Any undefined term here may be found in Harary [1].

A set D of vertices in a graph G is a dominating set of G if every vertex in V – D is adjacent to some vertex in D. The domination number $\gamma(D)$ of G is the minimum cardinality of a dominating set of G. A recent survey of $\gamma(G)$ can be found in Kulli [2].

Among the various applications of the theory of domination that have been considered, the one that is perhaps most often discussed concerns a communication network. To minimize the direct communication links in the network, in [4] Kulli and Janakiram introduced the concept of the nonbondage number as follows:

The nonbondage number $b_n(G)$ of a graph G is the maximum cardinality of all sets of edges $X\subseteq E$ for which $\gamma(G – X) = \gamma(G)$.

Let uv be an edge of G. Then u and v dominate each other. Further u strongly dominates v if deg u ≥ deg v. A set $D\subseteq V$ is strongly dominating set if every vertex v in V – D is strongly dominating by some u in D. The strong domination number $\gamma_s(G)$ of a graph G is the minimum cardinality of a strong dominating set, see [5].

The strong nonbondage number $b_{sn}(G)$ of a nonempty graph G is the maximum cardinality among all sets of edges $X\subseteq E$ for which rs (G-X) = rs (G), see [6]. The strong bondage number $b_s(G)$ of G is the minimum cardinality among all sets of edges $X\subseteq E$ for which $\gamma_s(G – X) > \gamma_s(G)$, see [8].

II. Exact Values of $b_n(G)$ for Some Standard Graphs

Proposition 1. If P_p is a path with $p\geq 4$ vertices, then

$$b_n(P_p)=\left\lceil \frac{p}{3}\right\rceil-1.$$

Proposition 2. If Cp is a cycle with $p\geq 3$ vertices, then

$$b_n(C_p)=\left\lceil \frac{p}{3}\right\rceil.$$

Proposition 3. If K_p is a complete graph with $p\geq 3$ vertices, then

$$b_n(K_p)=\frac{(p-1)(p-2)}{2}.$$

Proposition 4. If $K_{m,n}$ is a complete bipartite graph, then
$b_n(K_{m,n}) = mn – m – n + 2$.

Proposition 5. If Wp is a wheel with $p\geq 4$ vertices, then

$$b_n(W_p)= p-1.$$

III. Nonbondage Number

Theorem A [4]. For any graph G,

$$b_n(G) = q – p + \gamma(G) \qquad (1)$$

Theorem 1. For any tree T,
$b_n(T) = \gamma(T) – 1$.

Proof: This follows from (1) and for a tree T, q = p – 1.

Theorem 2. For any unicyclic graph G,
$b_n(G) = \gamma(G)$.

Proof: This follows from (1) and for any unicyclic graph G, p = q.

Theorem B [6]. For any graph G,
$b_n(G) \leq b_{sn}(G)$.

Theorem C [6]. For any tree T,

$$b_{sn}(T)\leq\left\lceil \frac{4(p-2)}{2}\right\rceil.$$

Theorem 3. For any tree T,

$$b_n(T)\leq\left\lceil \frac{4(p-2)}{2}\right\rceil.$$

Proof: This follows from Theorem B and Theorem C.

Theorem D [9]. For any graph G,

$$\gamma(G)+\gamma(\bar{G})\leq p+1$$

In the following theorems, we establish Nordhaus-Gaddum type results.

Theorem 4. For a graph G and its complement $\bar{G}$,

$$b_n(G)+b_n(\bar{G})\leq\frac{(p-1)(p-2)}{2}.$$

Proof: By Theorem A,

$$b_n(G)+b_n(\bar{G})= q-p+\gamma(G)+\bar{q}-p+\gamma(\bar{G})$$

$$=(q+\bar{q})-2p+\gamma(G)+\gamma(\bar{G})$$

$$\leq\frac{p^2-5p}{2}+p+1$$

or $b_n(G)+b_n(\bar{G})\leq \frac{(p-1)(p-2)}{2}$.

Theorem E[4]. For any graph G,
bn(G) ≤ q – Δ(G).

We now give another proof of Theorem 4.

Proof: By Theorem E,

$b_n(G) \leq q - \Delta(G) \leq q - \delta(G)$.

and $b_n(\bar{G})\leq \bar{q}-\Delta(\bar{G})$.

Thus $b_n(G)+b_n(\bar{G})\leq q-\delta(G)+\bar{q}-\Delta(\bar{G})$

$=q+\bar{q}-\left(\Delta(\bar{G})+\delta(G)\right)$

or $b_n(G)+b_n(\bar{G})\leq \frac{p(p-1)}{2}-(p-1)$

or $b_n(G)+b_n(\bar{G})\leq \frac{(p-1)(p-2)}{2}$

Theorem F[10]. If G and $\bar{G}$ are connected, then $\gamma(G)+\gamma(\bar{G})\leq p$ with equality holds if and only if G is P_4.

Theorem 5. If G and $\bar{G}$ are connected, then

$$b_n(G)+b_n(\bar{G})\leq \frac{p(p-3)}{2}.$$

Furthermore, equality holds if and only if G=P_4.

Proof: By Theorem A,

$$b_n(G)+b_n(\bar{G})= q+\bar{q}-2p+\gamma(G)+\gamma(\bar{G})$$

$$=\frac{p^2-5p}{2}+\gamma(G)+\gamma(\bar{G})$$

By Theorem F,

$$b_n(G)+b_n(\bar{G})\leq \frac{p^2-5p}{2}+p$$

or $b_n(G)+b_n(\bar{G})\leq \frac{p(p-3)}{2}$.

By Theorem F, equality holds if and only if G=P_4.

Theorem 6. If T and $\bar{T}$ are connected, then
$b_n(T)+b_n(\bar{T})\leq p-2$.

Furthermore, equality holds if and only if G=P_4.

Proof: From Theorem 1,

$b_n(T) = \gamma(T) - 1$

and $b_n(\bar{T})=\gamma(\bar{T})-1$.

Thus $b_n(T)+b_n(\bar{T})=\gamma(T)-\gamma(\bar{T})-2$.

By Theorem F, $\gamma(T)+\gamma(\bar{T})\leq p$.

Thus $b_n(T)+b_n(\bar{T})\leq p-2$.

By Theorem F, equality holds if and only if G is P_4.

Theorem G[11]. For any graph G,
χ(G) ≤ Δ(G)+1.

We establish a relation between the nonbondage number $b_n(G)$ and the chromatic number χ(G).

Theorem 7. For any graph G,
$b_n(G) + \chi(G) \leq q + 1$. (2)
and this bound is sharp.

Proof: By Theorem E, $b_n(G) \leq q - \Delta(G)$ and by Theorem G, χ(G) ≤ Δ(G)+1. Thus (2) holds.

The complete graphs K_p, p≥3, achieve this bound.

Theorem H[4]. Let G be a unicyclic graph. If $\gamma(G)=\frac{p}{2}$, then $b_n(G) \geq \Delta(G)$.

The following result involving the chromatic number χ(G) gives a lower bound for $b_n(G)$.

Theorem 8. Let G be a unicyclic graph. If $\gamma(G)=\frac{p}{2}$, then
$\chi(G) - 1 \leq b_n(G)$ (3)

Proof: By Theorem H, $\Delta(G) \leq b_n(G)$ and by Theorem G, χ(G) ≤ Δ(G)+1. Thus (3) holds.

Theorem I. For any graph G, κ(G) ≤ δ(G).

Theorem 9. For any graph G,
$b_n(G)+\kappa(G)\leq q$ (4)

Proof: By Theorem E, $b_n(G) \leq q - \Delta(G)$ and by Theorem I, κ(G) ≤ δ(G). Thus (4) holds.

The cycle C_3 achieves this bound.

IV. Strong Nonbondage Number

Theorem J[4]. For any connected graph G,

$$\frac{\text{diam}(G)-2}{3}\leq b_n(G).$$

Theorem 10.For any connected graph G,

(diam(G) - 2) / 3 < or = b_{sn} (G)

Proof: This follows from Theorem B and Theorem J.

We give a simple proof of the following theorem.

Theorem 11[6]. Let G be a unicyclic graph. If $\gamma(G)=\frac{p}{2}$, then $b_{sn}(G) \geq \Delta(G)$.

Proof: This follows from Theorem H and Theorem B.

Theorem K[4]: For any graph G, $b(G) \leq b_n(G)+1$.

The following result involving the bondage number gives a lower bound for bsn(G).

Theorem 12. For any graph G,
$b(G)-1 \leq b_{sn}(G)$. (5)

Proof: By Theorem B, $b_n(G) \leq b_{sn}(G)$ and by Theorem K, $b(G) \leq b_n(G)+1$. Thus (5) holds.

Theorem L[6]. For any graph G,
$b_{sn}(G) \leq q - \Delta(G)$.

We establish a relation between the strong nonbondage number $b_{sn}(G)$ and the chromatic number χ(G).

Theorem 13 For any graph G,
$b_{sn}(G) + \chi(G) \leq q + 1$ (6)
and this bound is sharp.

Proof: By Theorem L, $b_{sn}(G) \leq q - \Delta(G)$ and by Theorem G, χ(G) ≤ Δ(G) +1. Thus (6) holds.

ISSN : 2347 - 8446 (Online)
ISSN : 2347 - 9817 (Print)

The path P_3 achieves this bound.

Theorem 14. Let G be a unicyclic graph. If $\gamma(G)=\frac{p}{2}$, then

$\chi(G) - 1 \leq b_{sn}(G)$ (7)

and this bound is sharp.

Proof: By Theorem 10, $\Delta(G) \leq b_{sn}(G)$ and by Theorem G, $\chi(G) \leq \Delta(G) +1$. Thus (7) holds.

We obtain a relation between the strong bondage number $b_s(G)$ and the chromatic number $\chi(G)$.

Theorem M[6]. For any graph G,

$b_s(G) \leq q - \Delta(G) +1$.

Theorem 15. For any graph G,

$b_s(G) + \chi(G) \leq q + 2$.

Proof: This follows from Theorem G and Theorem M.

Theorem 16. For any graph G,

$b_{sn}(G) + \kappa(G) \leq q$ (8)

Proof: By Theorem K, $b_{sn}(G) \leq q - \Delta(G)$ and by Theorem I, $\kappa(G) \leq \delta(G)$. Thus (8) holds.

Theorem 17. For any graph G,

$b_s(G) + \kappa(G) \leq q +1$ (9)

Proof: By Theorem M, $b_s(G) \leq q - \Delta(G)+1$ and by Theorem I, $\kappa(G) \leq \delta(G)$. Thus (9) holds.

V. Bondage Number

We establish Nordhaus-Gaddum type results.

Theorem 18. For graphs G and $\overline{G}$,

$$b(G)+b(\overline{G}) \leq \frac{(p-1)(p-2)}{2}+2.$$

Proof: By Theorem J,

$$b(G)+b(\overline{G}) \leq b_n(G)+b_n(\overline{G})+2.$$

Thus

$$b(G)+b(\overline{G}) \leq \frac{(p-1)(p-2)}{2}+2$$, by Theorem 4.

Theorem 19. If G and $\overline{G}$ are connected, then

$$b(G)+b(\overline{G}) \leq \frac{p(p-3)}{2}+2$$

and this bound is sharp.

Proof: By Theorem K,

$$b(G)+b(\overline{G}) \leq b_n(G)+b_n(\overline{G})+2.$$

Thus by Theorem 4,

$$b(G)+b(\overline{G}) \leq \frac{p(p-3)}{2}+2.$$

The path P_4, achieves this bound.

Theorem N[4]. For any graph G,

$b(G) \leq q - \Delta(G) + 1$.

We obtain a relation between the bondage number b(G) and the chromatic number $\chi(G)$.

Theorem 20. For any graph G,

$b(G) + \chi(G) \leq q +2$ (10)

and this bound is sharp.

Proof: By Theorem N, $b(G) \leq q - \Delta(G) + 1$ and by Theorem G, $\chi(G) \leq \Delta(G)+1$. Thus (10) holds.

The path P_2 and the cycle C_3 achieve this bound.

We obtain a relation between the bondage number b(G) and the connectivity $\kappa(G)$.

Theorem 21. For any graph G,

$b(G) + \kappa(G) \leq q +1$

and this bound is sharp.

Proof: By Theorem M, and by Theorem I, we have

$b(G) + \kappa(G) \leq q - \Delta(G) + 1 + \delta(G)$

or $b(G)+\kappa(G) \leq q+1$, since $\delta(G)-\Delta(G) \leq 0$.

The cycle C^3 achieves this bound.

VI. Open Problems

We suggest the following problems for further study.

Problem 1. By Theorem 7,

$b_n(G) + \chi(G) \leq q +1$.

Characterize the corresponding extremal graphs.

Problem 2. By Theorem 9,

$b_n(G) + \kappa(G) \leq q$.

Characterize the corresponding extremal graphs.

Problem 3. By Theorem 13,

$b_{sn}(G) + \chi(G) \leq q +1$.

Characterize the corresponding extremal graphs.

Problem 4. By Theorem 15,

$b_s(G) + \chi(G) \leq q +2$.

Characterize the corresponding extremal graphs.

Problem 5. By Theorem 17,

$b_{sn}(G) + \kappa(G) \leq q$.

Characterize the corresponding extremal graphs.

Problem 6. By Theorem 18,

$bs(G) + \kappa(G) \leq q +1$.

Characterize the corresponding extremal graphs.

Problem 7. By Theorem 20,

$b(G) + \chi(G) \leq q +2$.

Characterize the corresponding extremal graphs.

Problem 8. By Theorem 21,

$b(G) + \chi(G) \leq q +1$.

Characterize the corresponding extremal graphs.

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